MINOR STUDIES FROM THE PSYCHOLOGICAL LABORATORY OF CORNELL UNIVERSITY

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XL. ON THE CALCULATION OF AN ASSOCIATIVE LIMEN

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Fundamentally the problem of the 'measurement of memory' is a counterpart of the problem of the measurement of sensation. In the one we seek a correlation between the 'strength' of an 'association' and the magnitude of some one of its conditions (*e.g.*,number of repetitions) ; in the other we wish to correlate a series of sensory impressions with a series of stimuli. In the psychophysics of sensation we may turn, in the first place, to the notion of a threshold. We can not, on the side of sensation, say how much more than adequately determined a present sensation or sense-difference is, or how much less than adequately determined an absent sensation or sense-difference is. We can state with assurance only that the sensation or sense-difference is or is not present. But if there were a stimulus just barely adequate to a given sensation or sense-difference, could determine it and call it a "threshold." As a matter of fact, however, no sense-organ remains constantly disposed for stimulation. Hence we are forced to substitute for the crude notion of the threshold that of the statistical limen, which is the stimulus-value that is just as often adequate as inadequate to the given sensation or sense-difference.[1] In like manner, in the measurement of memory we can not tell, when a given association is realized, how much more than adequate to reproduction it is, or, when it is not realized, how much it lacks of being adequate; but we can tell whether or not the association is realized. The determination of a threshold for memory appears to be the obvious 'way out;' but there are no fixed conditions that are always just adequate to reproduction, for the nervous system is ever varyingly disposed for memorial impression and reproduction, just as it is for sensory impression. Given a certain number of repetitions, let us say, then some associations are realized and others not. In such a case the proper measure seems to be the memorial limen, *i.e.*, the number of repetitions (or other measure) that produces just as many 'right associates' as not.[2]

[1] *Cf.* E. G. Boring, A Chart of the Psychometric Function, this JOURNAL, 28, 1917, 465; esp. p. 469 (note) and p. 470.

[2] The analogy between the problems of memory and those of the psychophysics of sensation is implicit in discussions of the memory-methods; *cf.* H. Ebbinghaus, *Memory,* 1913 (1885), Chap. II. Here and elsewhere the concept of the threshold is implied, although it is not explicitly formulated. A. Jost, *Zeitschr. f. Psychol.,* 14, 1897, 436ff., esp. 455-459, bases his discussion of right associates upon the notion of a *Treffergrenze.* So far as we are aware, no one, however, has attempted seriously to employ the thorough-going procedures of psychophysics to the problem of memory.

If we are to determine an associative limen we must use a method of partial mastery. Complete mastery would mean that every association was more than critically adequate to reproduction, and that we could not therefore interpolate the limen, the point of most frequent critical adequacy.[3] A 'constant method' (*Konstanzmethode*), utilizing the procedure of the method of constant stimuli, is, at the present time, the natural one to adopt. We may determine for different numbers of repetitions (say) the percentages of right associates, plot these percentages as a psychometric (mnemometric) function, and interpolate, as the limen, the point where the frequency of reproduction should be 50%. The *Treffermethode* lends itself readily to such treatment, although other methods of partial mastery, such as the methods of retained members and of prompting, can also be dealt with in this manner.

In adopting the 'constant method' we have still to decide what hypothesis of the psychometric function we shall use. Undoubtedly in an untried field an indifferent hypothesis[4] is ordinarily safest. The inexact results of memory experiments, however, make it desirable that the gross variations of the psychometric functions be smoothed off, under the principle of least squares, in accordance with some definite assumption. The *phi-gamma* hypothesis recommends itself for two reasons. In the first place, since it summarizes satisfactorily the psychophysical relationship in many cases of sensation, it is reasonable that we should tentatively attempt its extension to other psychophysical problems. In the second place, it is *a priori* the proper psychometric function, provided that the variations of nervous disposition are chance variations, and provided we are sure that our unit of measurement on the physical side is not merely a physical unit but actually a unit of *stimulus* or of some other condition of nervous impression.[5] We have adopted the *phi-gamma* hypothesis, thus *tentatively*, in this study.

Whether the 'mnemometric function' be the *phi-gamma* function or not, it seems obvious that it must have, in general, the same properties; that is to say, it must approach zero and unity gradually, and have its maximal slope at an intermediate point. From such a premise important conclusions affecting the value of the memory-methods follow. If the 'mnemometric function' is asymptotic to the 100%-line, complete mastery must be infinitely improbable. In any case, complete mastery, when seemingly achieved, is presumably due to the fewness of the cases, and its measure is exceedingly unreliable. Theoretically the weight, P, would be 0. On the other hand, half-mastery must have relatively great, perhaps maximal, reliability. Hence it follows in general (*a*) that a measure of complete mastery is less reliable than a measure of partial mastery, and that this difference works to the disadvantage of the method of complete mastery; (*b*) that percentages near 50% are probably more reliable in a method of partial mastery than percentages remote from 50%; and (*c*) that

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[3] See foot-note 1.

[4] See, for example, F. M. Urban, *Psychol. Rev.*, 17, 1910, 232ff; *Arch. f. d. ges. Psychol.*, 15, 1909, 335ff.

[5] Boring, *op. cit.*, the whole discussion, but esp. p. 469. The assumption of Boring's paper is that the centimeter is a true unit of stimulus; that every centimeter disposes by an equal amount toward the impression 'two;' that 0.8 cm., for example, is twice as effective as 0.4, and 1.6 twice as effective as 0.8.

percentage-differences in one part of the percentile scale (*e.g.*, 90% to 95%) are presumably not equivalent, in a method of partial mastery, to percentage-differences in another part of the percentile scale (*e.g.*, 55% to 60%), and that the two can not therefore be compared. Measures of reliability and comparisons of percentage-differences become possible as soon as a definite hypothesis has been fixed upon, as, for example, the *phi-gamma* hypothesis.

We have laid down a theoretical programme for the investigation of memory. As a rough preliminary test, we have undertaken the determination of two associative limens, using the experimental procedure of the method of right associates and the statistical treatment of the method of constant stimuli.

We had two observers, Miss M. Kincaid (*K*) and Mrs. H. D. Williams (*W*), both with a little previous experience in memory experiments. *K* is an ingenious learner; we had, after preliminary trials, to give her additional instructions, in the effort to prevent her from making punning associations upon the nonsense syllables. She never however, acquired the ability to maintain a constant disinterested attitude toward the series.

We used throughout series of ten nonsense syllables. We had in every series five short and five long vowels (a, e, i, o, u), indicating the long vowel by a diacritical mark. In all other respects we followed Gamble's modification of Müller and Schumann's rules.[6] The order of syllables in the test series was always different from the order in the presentation and different in different tests.

We adopted number of repetitions as a measure of associative strength. Each observer learned 50 ten-syllable series. At the first session five series were presented, every one with a different number of repetitions; at the second session five more series, with the same numbers of repetitions, and so on. There were ten such sessions in all. The order in which the five different numbers of repetitions were given was changed at every session, so that every number of repetitions came equally often (twice) at every place during the session (*i.e.*, twice in first, in second, in third, in fourth, and in last place; ten times in all in the ten sessions; once in every session).

The observers were practised for two weeks before the beginning of the regular series. The ten sessions came at the same hour of the day for each observer, five sessions on five consecutive days of one week, and five sessions on five days of the next week. The work was performed in the Cornell Summer Session, 1917.

The syllables were typewritten, and presented stepwise on the Spindler and Hoyer memory-apparatus at the rate of one syllable in 0.75 sec. The test-series for reproduction was presented 15 sec. after the conclusion of the learning-series. Within a single session every series was begun 10 min. after the beginning of the preceding series. There was thus an interval between successive series of about 5 min.

The general instruction was as follows: "After the signals 'ready,' 'now,' there will be presented to you successively a series of nonsense syllables. This series will be on the revolving drum of the machine and you will see one syllable at a time. You are to fixate the left-hand portion of the window and pronounce the syllables in trochaic rhythm as they appear. Give full and equal attention to every syllable. After a number of repetitions the window will be closed at the left

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[6] E. A. McC. Gamble, *Psychol. Rev. Monog.*, No. 43, 1909, 18ff.

and the first members of the pairs will be presented at the right. Then after the signal, 'now,' you are to fixate the right-hand portion of the window, and when the syllable appears pronounce as quickly as possible the syllable which followed it in the series. After you have pronounced it write it down. If you do not know the syllable which followed it, say, 'Don't know,' and put a dash on your paper."

When the experiment was completed we had presented fifty pairs of syllables for every one of five different numbers of repetitions. We scored the associates as right or half-right, and came out with even (non-fractional) percentages of right associates for every number of repetitions. The numbers of repetitions used for each observer were so determined in the preliminary experiments as to give percentages on either side of 50%. The actual numbers of repetitions used (r) and the resulting percentages of right associates (p) are as follows:

For K:	r	=	1	2	3	4	5
	p	=	.36	.45	.72	.69	.66
For W:	r	=	2	5	8	11	14
	p	=	.37	.59	.84	.83	.88

From these data we computed with the aid of Urban's tables (since we had arranged our procedure to give the even percentages which the tables require), the associative limens:

For K: $h = .1534$, $L = 2.1029$ repetitions.
For W: $h = .0920$, $L = 3.4244$ repetitions.

Considered in relation to the *phi-gamma* hypothesis, the data are manifestly unsatisfactory. Both sets show the last three percentages grouped closely together, both sets show inversions, and K's maximum occurs for the middle number of repetitions. We can tell how closely the hypothesis is approximated if we take the sums of the squares of the deviations of the theoretical percentages (the most probable values under the hypothesis) from the actual percentages,[7] Σd^2. The magnitude of this measure is dependent, however, upon the number of observation-equations; we get an independent value if we divide Σd^2 by the excess of the number of observation-equations ($M, = 5$ in these two experiments) over the number of normal equations ($\mu, = 2$ for the *phi-gamma* hypothesis). If we take the square root of this quotient, we get for different cases values that are directly comparable one with another (since they are *per cents*) and that involve few digits. The resulting measure of approximation to the hypothesis we may call[8]

$$\varepsilon = \sqrt{\frac{\Sigma d^2}{M - \mu}}$$

By computation[9] we get

For K: $\varepsilon = .101$
For W: $\varepsilon = .078$

The maximal individual value that Urban found for the *phi-gamma*

[7] *Cf.* Urban, *Psychol. Rev.*, 17, 258.

[8] *Cf.* W. W. Johnson, *Theory of Errors and Methods of Least Squares*, 1915, 108f.

[9] See Boring, this JOURNAL, 28, 1917, 284, 288, 291f.

hypothesis with lifted weights is $\varepsilon = .053$; his maximal average value is $\varepsilon = .044$; his minimal average value is $\varepsilon = .031$; his minimal individual value is $\varepsilon = .012$.[10] It appears then that our results do not satisfy the *phi-gamma* hypothesis nearly so satisfactorily as do Urban's results with lifted weights.

This state of affairs does not necessarily mean that the *phi-gamma* hypothesis is incorrect. It would come about if successive repetitions became progressively less effective for association (as they do for retention). In such a case, however, the repetition could not be taken as a unit of associative strength; we should have to determine what function of the number of repetitions the effective associative condition ('associative unit') is, just as in the psychophysics of sensation we must know what function of the physical unit of our stimulating instrument the effective value of stimulus is.[11]

We can not turn to the literature for the form of this function. The method of complete mastery gives it for retention,[12] but it can not give it for learning, since this method assumes the repetition as its unit. No method of partial mastery[13] can give it, because every such method uses a *percent* of complete mastery as a unit, and equal percentile ranges can not represent equal differences of the effective condition of association on any other assumption than the very improbable one that the "mnemometric" function is a straight line.

Let us call the measure of the effective condition of association, a. then $a = f(r)$, but the f-function is unknown. We began by assuming $a = r$, and found that we did not approximate the *phi-gamma* function closely. We can not from our data determine the natures of both this f-function and of the "mnemometric" function; but we may assume either one and study the other.

Suppose now that, in addition to our original assumption, $a = r$, we assume tentatively that $a = \log r$, that $a = \sqrt{r}$, and that $a = \sqrt[3]{r}$.[14] All three functions give progressively smaller increments of a for every additional r. The square root function is less eccentric than

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[10] Computed from *Psychol. Rev.*, 17, 258.

[11] *E.g.*, in using Fechner's sound pendulum, the unit of the instrument is the degree, and the angle, Θ, is measured in degrees; but successive degrees do not represent equal increments of auditory stimulus; the unit of stimulus is $\sin^2 \frac{1}{2}\Theta$. In the same manner we may find that the effective condition of association (associative "stimulus") is some other function of the number of repetitions than a direct proportion. *Cf.* foot-note 5 above.

[12] Ebbinghaus, *op. cit.*, 52-61.

[13] W. G. Smith, *Psychol. Rev.*, 3, 1896, 21-31, gives such a function obtained by a method of retained members; E. Ephrussi, *Exper. Beiträge zur Lehre vom Gedächtnis*, 1904. 109-121, by the method of promptings; C. Knors, *Arch. f. d. ges. Psychol.*, 17, 1910, 297-361, by the *Treffermethode* and an *Aussagemethode*.

[14] It might be argued that the general logarithmic formula is not $a = \log r$, but $a = k \log r + b$, where k and b are constants. We may, however, let $k = 1$, if we do not care about the size of the associative unit in which a is measured; and we may let $b = 0$, if we do not care for what origin we measure a. Having determined our limen on these assumptions, we may alter *ex post facto* the size of the unit or the point of origin in accordance with any further consideration that may influence us.

The same argument applies to the other functions of r.

the logarithmic function, the cube root more so. We may use these values of a as abscissa-values, and compute h and the limen. We can not use Urban's tables, because our abscissa-values are no longer equidistant. We must multiply out. The limen and h come out in terms of the a-units, but the limen (not h) can be changed back into repetitions by taking the number whose logarithm it is, or by squaring it, or by cubing it, as the case may be. The results of these computations we give below. We add for W the case of $a = \sqrt[100]{r}$ to show that there is no great difference to be expected from taking higher powers.

	a	$= r$	$\log r$	$\sqrt{r}$	$\sqrt[3]{r}$	$\sqrt[100]{r}$
Limen in reps..	K	2.103	1.862	1.994	1.948	.....
	W	3.424	3.169	3.363	3.261	3.040
ε in %.........	K	.101	.082	.090	.087	....
	W	.078	.050	.055	.051	.054

It is obvious that the limen depends upon the hypothesis that we assume. Apparently, if $a = \sqrt[x]{r}$, then the greater the value of x the lower the value of the limen. (The limen is maximal when $x = 1$; minimal when $x = 3$ for K or 100 for W.) For $x = \log r$, we get lower values of the limen than for the cube root. We might decide which function, and hence which limen, to accept, by determining which approximates the *phi-gamma* hypothesis most closely. The values of ε show these approximations. They show that the square root satisfies the *phi-gamma* hypothesis better than the direct equation, $a = r$; the cube root is even better; improvement does not necessarily go on indefinitely, for the 100th root in the case of W is worse again; and the logarithmic function is still better than any of the root-functions tried. If we take Urban's work with lifted weights as our standard, then all the values of ε for K are too high. K was presumably too ingenious and erratic to give the steady results presupposed under the *phi-gamma* hypothesis. The values of ε for W are, however, comparable with Urban's. Two of them are less than his maximal value, .053. It appears, therefore, that, given the proper function that a is of r, together with consistent observation, we have reason to expect that the *phi-gamma* hypothesis will serve as a basis for the computation of memory-limens with a degree of reliability not very much less than the reliability with which the same hypothesis serves in the psychophysics of sensation.

We have sought to support this tentative conclusion by cases drawn from the literature of memory. For this purpose we selected the two more consistent of Knors' three adult observers, A and B; and we took for each of these observers the case where the *Treffermethode* was used with 10-syllable nonsense series. The cases are thus directly comparable with our own. In addition we took the average of Smith's results for a 10-syllable series obtained by a method of retained members, and Ephrussi's results for a 10-syllable series with Ebbinghaus as observer by the method of promptings.[15] Since the root

[15]Knors, *op. cit.*, 319; Smith, *op. cit.*, 26; Ephrussi, *op. cit.*, 111.

functions did not differ greatly with respect to ε from the logarithmic function in the preceding trials, we made these additional calculations merely for $a=r$ and $a=\log r$. The values of the limens and of ε for these new cases and for our own observers, K and W, are tabulated herewith.

	No. of observations	Limens in reps. r	Limens in reps. $\log r$	ε in % r	ε in % $\log r$
K	5	2.103	1.862	.101	.082
W	5	3.424	3.169	.078	.050
Knors-A	6	1.880	1.626	.105	.089
Knors-B	5	1.602	1.468	.056	.005
Ephrussi-Ebbinghaus	9	2.509	2.070	.072	.084
Smith (av.)	5	3.379	2.624	.064	.022

The table substantiates our former conclusions. The logarithmic formula gives always the lower limen and, with the exception of the Ephrussi-Ebbinghaus case, the closer approximation to the *phi-gamma* hypothesis. In Smith's average case, ε is less than Urban's minimal average (.031). Knors' observer B approximates the *phi-gamma* hypothesis more closely than any individual case of Urban's. (Urban's minimal ε is .012.)

The two hypotheses, *viz.*, that the "mnemometric" function is the *phi-gamma* function and that effectiveness for associative impression varies with the logarithm of the number of repetitions, work so well together that one tends naturally to jump to the conclusion that each has proved the other. And if we take certain extraneous facts into account, this conclusion is not necessarily the bad logic it would seem to be. We know, on the one hand, that the *phi-gamma* hypothesis does summarize certain facts of the nervous system which hold in the sphere of sensation, and that its properties reflect the property of chance dispositional variation which we have good reasons to believe is characteristic of nervous tissue. We have indirect indications, on the other hand, although no direct proof, that successive repetitions are progressively less effective for learning. Jost's law, for example, in its simplest interpretation suggests some such fact. If we can prove neither hypothesis alone, but if there are nevertheless initial presumptions in favor of each separately, and if in conjunction the two hypotheses constitute jointly an approximate summary of observed data, then each hypothesis gains support from the other; and, at any rate, the two together may be regarded as a single hypothesis adequate to the facts, until new data are discovered which will not conform with the compound assumption.

Unfortunately for theory, we have as yet very few data that bear upon the problem. Data obtained under an inconstant attitude, like K's, are scarcely admissible. If we accept such published results as there are upon their face value, then we do not always find that the logarithmic assumption, in conjunction with the *phi-gamma* hypothesis, is borne out. In general, so far as our analysis of the cases has gone, $a=\log r$ seems to give a smaller ε than $a=r$. We have seen that the Ephrussi-Ebbinghaus case is an exception; and we can find other exceptions. For example, in the 18-syllable series with Knors' observer B, $A=\log r$ is a better assumption than $a=r$, but $a=\sqrt{r}$ is better still. In the 18-syllable series with Knors' observer A, $a=r$ is an almost perfect hypothesis, much better than any other

that we tried.[16] Obviously a thorough-going study of compound hypotheses is needed. Such a study must, moreover, be made presumably upon data that are taken under more careful conditions of attitudinal control of the observer than are found in the ordinary memory experiment.

We conclude: that the problem of the 'measurement of memory,' like other psychophysical problems, can be solved most reliably and in terms that admit of the widest comparison by the determination of an associative limen; that a constant method, like the method of constant stimuli, is applicable; that such a method involves the determination of a mnemometric function, comparable with the psychometric function for sensation, and that there is reason to believe that the *phi*-function of *gamma* may prove to be as satisfactory an hypothesis in the field of memory as it has proved to be in the field of sensation; that the determination of the associative limen depends further, however, upon the use of some proper measure of the effective condition of association; that the repetition is probably not such a measure, and that the function that this measure is of the number of repetitions of a given material must constitute a second assumption of the method, at least in so far as it can not be determined independently; and finally that the two-fold assumption—that the mnemometric function is the *phi*-function of *gamma* and that the effective condition of association varies with the logarithm of the number of repetitions—appears to be in approximate accordance with the facts, although the experimental data are insufficiently reliable and show some exceptions.

We wish to emphasize the tentative nature of this study; we have sought rather to indicate a method than to make a discovery. Our work, we think, should be done over again under the most constant attitudinal conditions obtainable in an observer. It should be paralleled further by a study in which length of series is the variable in terms of which the limen is to be stated. The method could then be extended to retention. Meanwhile a thorough-going working over of the results already in the literature, with an indifferent hypothesis as well as with the *phi*-function of *gamma*, might prove valuable.

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[16] There is a simpler way of dealing with hypotheses of the value of a than the one described: it lacks, however, the advantage of giving the values of ε used above. In a given case one plots against values of r the corresponding values of γ obtained from the p for every r. Under the *phi-gamma* hypothesis, γ must be a direct measure of a. Hence, if r is a direct measure of a, we get a straight line. If the curve as plotted does not approach a straight line, we can try plotting γ against $\log r$, $\sqrt{r}$, and so on, and select as the best hypothesis the nearest approach to a straight line. It is often easy to make a decision by inspection of the curves as actually plotted on graph paper. In a careful investigation the curves could be adjusted by the method of least squares, with every γ weighted by the corresponding P for the *phi-gamma* hypothesis.